

Decay process of quantum open system at finite-temperature

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Starting from the formal solution to the Heisenberg equation, we revisit an universal model for a quantum open system with a harmonic oscillator linearly coupled to a boson bath. The analysis of the decay process for a Fock state and a coherent state demonstrate that this method is very useful in dealing with the problems in decay process of the open system. For finite temperature, the calculations of the reduced density matrix and the mean excitation number for the open system show that an initial coherent state will evolve into a temperature-dependant coherent state after tracing over the bath variables. Also in short-time limit, a temperature-dependant effective Hamiltonian for the open system characterizes the decay process of the open system.

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I. INTRODUCTION

Actually, no quantum system can be isolated from its surrounding environment completely [1, 2]. When it is coupled to the environment, the system (also named open system) energy dissipates into the environment irreversibly. There exists two approaches to deal with the decay process of the open system, i.e., the system plus bath model [3] and effective Hamiltonian model [4, 5]. In a simple model of a harmonic oscillator interacting with a bath of many harmonic oscillators, the relation between these two approaches is clarified by Yu and Sun [6, 7].

In quantum optics and macroscopic quantum phenomena, the mesoscopic or macroscopic states can be represented by coherent states. Due to the quantum fluctuation and thermal fluctuation of the bath, it is very important to study how the open system evolves with an initial Fock state [8] and an initial coherent state [9], especially at finite temperature.

Usually, the master equation method is adopted to treat the decay process of the open system [10], time evolution of the reduced matrix for the open system can be solved analytically and numerically. In this paper, starting from the formal solution to the Heisenberg equation, we can construct the reduced density matrix for the open system, this method has been applied to deal with some problems in quantum open system [6, 7, 11–14]. Here, an exactly solvable model of one boson interacting with a bath of many bosons in rotating-wave approximation is used to analyze the decay process of the open system.

The paper is organized as follows. In sec.II, we describe an universal model for quantum open system, a single mode boson interacting with a bath of many bosons through system-bath coupling. In sec.III, we revisit the decay process of a Fock state and a coherent state at zero temperature. In sec.IV, we study the finite-temperature decay process of the open system. Finally, we summarize our results.

II. SYSTEM-BATH MODEL

The quantum decay process is characterized with the vanishing of diagonal elements of the time evolution of reduced density matrix for an open system. The simplest model is adopted that the composite system consists of the system (a single mode boson) interacting with the bath of multi-mode bosons. Using the method in Refs. [6, 7, 11–14], we can construct the reduced density matrix and calculate the mean excitation number for the open system.

The model Hamiltonian for this composite system is

$$H = \hbar\omega_b b^\dagger b + \sum_j \hbar\omega_j a_j^\dagger a_j + \sum_j \xi_j (b a_j^\dagger + b^\dagger a_j), \quad (1)$$

where b (a_j) and b^\dagger (a_j^\dagger) are bosonic annihilation and creation operators for the system (bath), and the ξ_j are complex coupling constants parameterized via the spectral density function $J(\omega)$, i.e.,

$$J(\omega) = \sum_j |\xi_j|^2 \delta(\omega - \omega_j). \quad (2)$$

Here we have applied rotating-wave approximation on the model Hamiltonian H .

Now we can use the time evolution of operator in Heisenberg picture to construct a wavefunction (or a reduced density matrix). These well-known solutions of Heisenberg operators are given in Ref.[11]

$$b(t) = u(t) b(0) + \sum_j v_j(t) a_j(0),$$

$$a_j(t) = e^{-i\omega_j t} a_j(0) + u_j(t) b(0) + \sum_s v_{j,s}(t) a_s(0).$$

Where some coefficients in the above equation are

$$\begin{aligned} u(t) &= e^{-\frac{\gamma}{2}t} e^{-i\omega_b t}, \\ v_j(t) &= \xi_j e^{-i\omega_j t} \frac{e^{-\frac{\gamma}{2}t} e^{-i(\omega_b - \omega_j)t} - 1}{\omega_b - \omega_j - i\frac{\gamma}{2}}, \\ u_j(t) &= \xi_j^* e^{-i\omega_j t} \frac{e^{-\frac{\gamma}{2}t} e^{-i(\omega_b - \omega_j)t} - 1}{\omega_b - \omega_j - i\frac{\gamma}{2}}. \end{aligned} \quad (3)$$

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When the spectral function of the bath is assumed as

$$J(\omega) = \frac{\gamma}{\pi}, \quad (4)$$

the calculations show that

$$\sum_j |v_j(t)|^2 = 1 - e^{-\gamma t}. \quad (5)$$

and $|u(t)|^2 + \sum_j |v_j(t)|^2 = 1$ which maintains the commutation relation $[b(t), b^\dagger(t)] = 1$.

III. ZERO-TEMPERATURE DECAY PROCESS

In quantum optics, the study of Fock state and coherent state is very important. At zero temperature, the decay of the Fock state and coherent state is presented with master equation method in Refs. [8, 9]. Here, using the method in Refs. [6, 7, 11–14], we revisit the decay of a Fock state and a coherent state.

A. Decay of a Fock state

The initial state of the composite system is assumed in

$$|\psi(0)\rangle = |n\rangle \otimes |\{0_j\}\rangle, \quad (6)$$

where n stands for the excitation number for the Fock state $|n\rangle$ and the bath is in vacuum state $|\{0_j\}\rangle$. The ground state of the composite system is $|0\rangle \otimes |\{0_j\}\rangle$, which is invariant under the operation of Unitary evolution operator $U(t)$, i.e., $U(t)|0\rangle \otimes |\{0_j\}\rangle = |0\rangle \otimes |\{0_j\}\rangle$. Then we can construct the time evolution of the composite system,

$$\begin{aligned} |\psi(t)\rangle &= U(t)|\psi(0)\rangle \\ &= \frac{(B^\dagger(t))^n}{\sqrt{n!}} |0\rangle \otimes |\{0_j\}\rangle. \end{aligned}$$

Where the operator $B(t)$ is defined as $B(t) = U(t)b(0)U^\dagger(t)$. Using the method in Ref.[14], we can replace i with $-i$ in Eq.(3) and get

$$B^\dagger(t) = u(t)b^\dagger(0) + \sum_j v_j(t)a_j^\dagger(0). \quad (7)$$

Tracing over the bath variables in density matrix for the composite system $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, the reduced density matrix for the open system is calculated as

$$\rho_s(t) = \sum_{m=0}^n P_m(t) |m\rangle\langle m|$$

where the probability in the state $|m\rangle$ for $m = 0 \dots n$ is

$$P_m(t) = \frac{n!}{(n-m)!m!} e^{-m\gamma t} (1 - e^{-\gamma t})^{n-m} \quad (8)$$

Here we have taken into account the relation

$$\langle\{0_j\}|\left(\sum_i A_i\right)^k \left(\sum_j A_j^\dagger\right)^l |\{0_j\}\rangle = k! (1 - e^{-\gamma t})^k \delta_{kl}, \quad (9)$$

where $A_j = v_j^*(t)a_j(0)$. According to the result in Eq.(8), we obtain the probability in the state $|n\rangle$, $P_n(t) = e^{-n\gamma t}$ (seen in Ref. [8]). Here the decay time of the open system is defines by

$$\tau^{-1} = n\gamma. \quad (10)$$

B. Decay of a coherent state

Similar as the decay of a Fock state, for an initial coherent state $|\alpha\rangle$, we construct the time evolution of wavefunction for the composite system,

$$\begin{aligned} |\psi(t)\rangle &= e^{\alpha B^\dagger(t) - \alpha^* B(t)} |0\rangle \otimes |\{0_j\}\rangle \\ &= |\alpha u(t)\rangle \otimes \prod_j |\alpha v_j(t)\rangle. \end{aligned}$$

Then the corresponding reduced density matrix is

$$\rho_s(t) = |\alpha u(t)\rangle\langle\alpha u(t)|.$$

Through the calculation of the mean excitation number for the open system,

$$\overline{N} = \text{Tr}(\rho_s(t)b^\dagger b) = |\alpha|^2 e^{-\gamma t},$$

the decay time of a coherent state $|\alpha\rangle$ is obtained (seen in Ref.[9]),

$$\tau^{-1} = \gamma. \quad (11)$$

When the bath is in excited state (coherent state $|\{\lambda_j\}\rangle$), the master equation method is not valid. Using our method, we can easily obtain the time evolution of wavefunction for the composite,

$$\begin{aligned} |\psi(t)\rangle &= e^{\alpha B^\dagger - \alpha^* B} |0\rangle \otimes \prod_j e^{\lambda_j A_j^\dagger - \lambda_j^* A_j} |0_j\rangle \\ &= |\mu(t)\rangle \otimes |\{\mu_j(t)\}\rangle. \end{aligned} \quad (12)$$

where the two coefficients in above equation are

$$\begin{aligned} \mu(t) &= \alpha u(t) + \sum_j \lambda_j u_j(t), \\ \mu_j(t) &= \alpha v_j(t) + \lambda_j e^{-i\omega_j t} + \sum_{s \neq j} v_{s,j}(t). \end{aligned}$$

IV. FINITE-TEMPERATURE DECAY PROCESS

In previous section, it shows that when the bath is prepared in a vacuum state, the quantum vacuum fluctuation of the bath will induce decay process of the open system. Here we will demonstrate the finite-temperature influence on decay process of the open system when the bath is in a thermal equilibrium state.

Initially, the density operator for the composite system is written as a direct product

$$\rho(0) = (|\psi(0)\rangle \langle \psi(0)|) \otimes \rho_B.$$

Here the initial state of the open system is assumed in a coherent state, $|\psi(0)\rangle = |\alpha\rangle$. In coherent-state representation, the bath at thermal equilibrium is described by the density operator

$$\rho_B = \prod_j \int \frac{d^2\lambda_j}{\pi\bar{n}_j} e^{-|\lambda_j|^2/\bar{n}_j} |\lambda_j\rangle \langle \lambda_j|,$$

where the mean excitation number in the j -th mode of the bath with the frequency ω_j is

$$\bar{n}_j = (e^{\beta\hbar\omega_j} - 1)^{-1}. \quad (13)$$

Using the result in Eq.(12), the reduced density matrix for the open system is calculated,

$$\rho_s(t) = \prod_j \left[\int \frac{d^2\lambda_j}{\pi\bar{n}_j} e^{-|\lambda_j|^2/\bar{n}_j} |\mu(t)\rangle \langle \mu(t)| \right]. \quad (14)$$

It can be used to characterize the decay process of the open system. Then the probability of the open system in the initial state $|\psi(0)\rangle = |\alpha\rangle$, i.e., diagonal elements of the reduced density matrix, can be obtained

$$\langle \psi(0) | \rho_s(t) | \psi(0) \rangle = \langle \alpha | (|\psi_s(t)\rangle \langle \psi_s(t)|) | \alpha \rangle.$$

Here the temperature-dependant wavefunction for the open system is obtained,

$$|\psi_s(t)\rangle = \Phi(T, t)^{-\frac{1}{2}} \left| \alpha \left((u(t) - 1) \Phi(T, t)^{-\frac{1}{2}} + 1 \right) \right\rangle. \quad (15)$$

It shows that at finite temperature the state of the system evolves into a coherent state $|\psi_s(t)\rangle$ according to a coherent state $|\alpha\rangle$ initially. In Eq.(15), the time-(temperature)-dependant term is denoted as

$$\Phi(T, t) = 1 + \sum_j \bar{n}_j |u_j(t)|^2. \quad (16)$$

Changing the sum into an integral in the above equation, we have

$$\sum_j \bar{n}_j |u_j(t)|^2 = \frac{\gamma}{2\pi} \int d\omega \frac{\left| e^{-\frac{\gamma t}{2} - i(\omega_b - \omega)t} - 1 \right|^2}{(\omega_b - \omega)^2 + \left(\frac{\gamma}{2}\right)^2} (e^{\beta\hbar\omega} - 1)^{-1}.$$

Here the term $(e^{\beta\hbar\omega} - 1)^{-1}$ is a slow-varying function of the frequency ω , thus we can take it out of the integral and get

$$\Phi(T, t) = 1 + \bar{n}_{th} (1 - e^{-\gamma t}),$$

where the mean excitation number with the frequency ω_b is $\bar{n}_{th} = (e^{\beta\hbar\omega_b} - 1)^{-1}$.

To demonstrate the decay process of the open system, we now calculate the mean excitation number for the open system. A decay process means that the mean excitation number reduces with the time increasing. Applying the time evolution of the wavefunction for the open system in Eq.(15), we can calculate the mean excitation number to characterize the decay process of the open system, i.e.,

$$\begin{aligned} \bar{N} &= \langle \psi_s(t) | b^\dagger b | \psi_s(t) \rangle \\ &= |\alpha|^2 \left| (u(t) - 1) \Phi(T, t)^{-1} + \Phi(T, t)^{-\frac{1}{2}} \right|^2. \end{aligned}$$

For low temperature ($\beta \rightarrow \infty$), we approximately have $\bar{n}_{th} \simeq e^{-\beta\hbar\omega_b}$. Then the mean excitation number of the open system becomes $\bar{N} \simeq |\alpha|^2 e^{-\gamma t}$. It shows that, in the low-temperature limit, the above mean excitation number \bar{N} exponentially decays as the time t increases and approximately does not depend the temperature T . That is to say, the vacuum fluctuations dominate the decay process of the open system.

For high temperature ($\beta \rightarrow 0$), we have $\bar{n}_{th} \simeq (\beta\hbar\omega_b)^{-1}$ and then the excitation number becomes

$$\bar{N} \simeq \beta\hbar\omega_b |\alpha|^2 (1 - e^{-\gamma t})^{-1} \propto \frac{1}{T}.$$

It shows that, in the high-temperature limit, the rise of the temperature will accelerate the decay of the open system.

Assume the system initially in a state $|\psi_s(0)\rangle$, it can be expanded into the superposition of a series of coherent states, i.e.,

$$|\psi_s(0)\rangle = \int \frac{d^2\alpha}{\pi} C_\alpha |\alpha\rangle, \quad (17)$$

where the amplitude of the probability in the state $|\alpha\rangle$ is $C_\alpha = \langle \alpha | \psi_s(0) \rangle$. Then the corresponding reduced density matrix will be

$$\rho_s(t) = \int \frac{d^2\alpha}{\pi} C_\alpha |\psi_{s,\alpha}(t)\rangle \int \frac{d^2\beta}{\pi} C_\beta^* \langle \psi_{s,\beta}(t) |, \quad (18)$$

where for $\xi = \alpha, \beta$, we have defined

$$|\psi_{s,\xi}(t)\rangle = \Phi(T, t)^{-\frac{1}{2}} \left| \xi \left((u(t) - 1) \Phi(T, t)^{-\frac{1}{2}} + 1 \right) \right\rangle.$$

In short-time limit, $\gamma t \ll 1$ and $\Phi(T, t) \simeq e^{\bar{n}_{th}\gamma t}$, the temperature-dependant wavefunction of the open system becomes

$$|\psi_{s,\xi}(t)\rangle = e^{-iH_{eff}t} |\xi\rangle. \quad (19)$$

According to the results in Eq.(18,19), the corresponding temperature-dependant reduced density matrix for the open system in Eq.(18) can be rewritten into

$$\rho_s(t) = e^{-iH_{eff}t} |\psi_s(0)\rangle \langle \psi_s(0)| e^{iH_{eff}t},$$

where the temperature-dependant effective Hamiltonian for the open system is obtained,

$$H_{eff} = \left(\omega_b - i\frac{\gamma}{2} \right) b^\dagger b - i\frac{1}{2}\overline{n_{th}}\gamma. \quad (20)$$

Consider the the open system initially in a Fock state, $|\psi_s(0)\rangle = |n\rangle$, the corresponding time evolution will be

$$|\psi_s(t)\rangle = e^{-iH_{eff}t} |n\rangle = e^{-in\omega_b t} e^{-\frac{1}{2}(\overline{n_{th}}+n)\gamma t} |n\rangle$$

and the mean excitation number becomes

$$\overline{N} \simeq n e^{-(\overline{n_{th}}+n)\gamma t} \quad (21)$$

in which the decay time of the open system is

$$\tau^{-1} = (\overline{n_{th}} + n)\gamma. \quad (22)$$

In a same fashion, consider the open system initially in a coherent state, $|\psi_s(0)\rangle = |\alpha\rangle$, the open system will evolve into a temperature-dependant coherent state

$$|\psi_\alpha(t)\rangle = e^{-iH_{eff}t} |\alpha\rangle = e^{-\frac{1}{2}\overline{n_{th}}\gamma t} \left| \alpha e^{-i(\omega_b - i\frac{\gamma}{2})t} \right\rangle.$$

Then the corresponding mean excitation number is calculated as

$$\overline{N} = |\alpha|^2 e^{-(\overline{n_{th}}+1)\gamma t} \quad (23)$$

with the decay time

$$\tau^{-1} = (\overline{n_{th}} + 1)\gamma. \quad (24)$$

The results in Eq.(22) and Eq.(24) show that thermal mean excitation number of the bath $\overline{n_{th}}$ accelerates the decay of the open system. When at zero temperature, $\overline{n_{th}} = 0$, the results in Eq.(22) and Eq.(24) come back to the results at zero temperature in Eq.(10) and Eq.(11).

V. CONCLUSIONS

In this paper, we have studied an universal model for a quantum open system with system-bath coupling. Using the method in Refs. [6, 7, 11–14], not master equation method, we revisit the decay process of the open system at zero temperature. Specially in short-time limit, a temperature-dependant effective Hamiltonian for the open system is obtained (seen in Eq.(20)). For finite temperature, it can characterize the decay process of the open system very well. The decay time shows that the temperature T will accelerate the decay process of the open system. In addition, a temperature-dependant wavefunction for the open system is easily obtained (seen in Eq.(15)), it shows that at finite temperature, the open system will evolve into a temperature-dependant coherent state from an initial coherent state. In summary, this method in Refs. [6, 7, 11–14] is very useful to deal with the problems of quantum open system.

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